These equations were used to compute the results in:

[1] K. Siwiak, “An Optimum Height for an Elevated HF Antenna” QEX May/June 2011.
[2] K. Siwiak, "What's the Optimum Height for an HF Antenna?” QST June 2011.
[3] K. Siwiak, “Optimum Height for an Elevated Communications Antenna”, DUBUS Magazine, Vol. 39, 3rd Quarter 2010, pp. 86-99.

HF propagation over spherical earth, With takeoff angle included Kai Siwiak June 2011

Constants: $Kr := 1.33$ earth refraction factor $Re := 6371009$ m

$Kr := 1.33$ $Ae := Re \cdot Kr$ m. earth radius, with refraction factor Kr

$c := 299.792458$

$Ae = 8.473 \times 10^{6}$

$rtd := \frac{180}{\pi}$ $dtr := \frac{1}{rtd}$ $TOL \equiv 10^{-14}$

$Hion := 100000$ Height of F layer is chosen results in a hop of 3240 km, or the ionospheric reflection point is about 1620 km away

$HiE := 100000$ E layer, for 10 MHz band and lower

$hrms := 3$ representative of 3.5m + 4.0m + 0m [street] in equal measures locale use 2 for sea; NOTE that hrms/2 is subtracted from Hant during calculations to adj for avg height

$Hant := 100$

Parameters:

Hion = height of ionosphere, m F layer
HiE = height of the E layer, m
Hb = height of antenna, m An initial test case antenna height, for Hb
Dhor = distance to horizon, m
eH = angle to horizon (negative)
e = elevation to ionosphere point relative to local horizontal vector: Skolnik's θ_d
e= 0 is |eH| above the radians
T = arrival angle = e, radians
Tion = incidence angle on ionosphere = G/Re+T, radians
γ = arc angle relative to earth center
γH = arc angle to radio horizon

DTr = Direct path distance from antenna to point on ionosphere
Rfi = Reflected path, ionosphere to refelction point
Rfd = Reflected path, Hb to reflection point
Re = Earth radius, m
G = Path on Earth arc from Hb to Hion projection on Earth

σ = ground conductivity, mho/m
εr = ground relative permittivity
hrms = ground rms roughness parameter, m

fMHz = operating frequency, MHz

Spherical earth geometry:

$$\mathrm{Dhor}(h) := \sqrt{2\cdot Ae\cdot h - h^2} \qquad \frac{\mathrm{Dhor}(\mathrm{Hant})}{1000} = 41.166 \qquad \frac{\mathrm{Dhor}(\mathrm{Hion})}{1000} = 1297.955$$

Maximum range to ionosphere: $\mathrm{Dmax} := \mathrm{Dhor}(\mathrm{Hion}) + \mathrm{Dhor}(\mathrm{Hant})$ $\qquad \mathrm{Dmax} = 1339121.939$

$$\gamma H(\mathrm{Hb}) := \mathrm{atan}\left(\frac{\mathrm{Dhor}(\mathrm{Hb})}{Ae}\right) \qquad \gamma H(\mathrm{Hant})\cdot\frac{180}{\pi} = 0.278$$

$$eH(\mathrm{Hb}) := \gamma H(\mathrm{Hb}) \qquad eH(\mathrm{Hant})\cdot\frac{180}{\pi} = 0.278$$

Hion>Hb, then:
U=angle(Ae+Hion, DTr)

$$\mathrm{sinU}(T,\mathrm{Hb},\mathrm{Hion}) := (Ae + \mathrm{Hb})\cdot\frac{\cos(T)}{(Ae + \mathrm{Hion})}$$

$$U(T,\mathrm{Hb},\mathrm{Hion}) := \mathrm{asin}(\mathrm{sinU}(T,\mathrm{Hb},\mathrm{Hion}))$$

Arc angle of path along earth:
=angle(Ae+Hion, Ae+Hb)

$$\gamma(T,\mathrm{Hb},\mathrm{Hion}) := \frac{\pi}{2} - T - U(T,\mathrm{Hb},\mathrm{Hion})$$

Exact direct path DTr in terms of take off angle T:

$$\mathrm{DTr}(T,\mathrm{Hb},\mathrm{Hion}) := \sin(\gamma(T,\mathrm{Hb},\mathrm{Hion}))\cdot\frac{(Ae + \mathrm{Hion})}{\cos(T)}$$

$$\mathrm{ang} := 0$$

Angle of max range: $\mathrm{angMin} := \mathrm{root}(\mathrm{DTr}(\mathrm{ang}\cdot\mathrm{dtr},\mathrm{Hant},\mathrm{Hion}) - \mathrm{Dmax},\mathrm{ang})$ $\qquad \mathrm{angMin} = -0.228$ deg

$$\mathrm{DTr}(\mathrm{angMin}\cdot\mathrm{dtr},\mathrm{Hant},\mathrm{Hion}) = 1339121.93860012$$

$$\mathrm{Dmax} = 1339121.93860012$$

Exact Distance along the ground arc:

$$G(T,\mathrm{Hb},\mathrm{Hion}) := Ae\cdot\gamma(T,\mathrm{Hb},\mathrm{Hion})$$

$$\frac{\mathrm{DTr}(2.3\cdot\mathrm{dtr},17,\mathrm{Hion})}{1000} = 1009.033$$

From Skolnik, Radar Handbook 2nd ed:

$$p(T,\mathrm{Hb},\mathrm{Hion}) := \frac{2}{\sqrt{3}}\cdot\sqrt{Ae\cdot(\mathrm{Hion} + \mathrm{Hb}) + \left(\frac{G(T,\mathrm{Hb},\mathrm{Hion})}{2}\right)^2}$$

$$\mathrm{zeta}(T,\mathrm{Hb},\mathrm{Hion}) := \mathrm{asin}\left[\frac{2\cdot Ae\cdot G(T,\mathrm{Hb},\mathrm{Hion})\cdot(\mathrm{Hion} - \mathrm{Hb})}{p(T,\mathrm{Hb},\mathrm{Hion})^3}\right] \qquad \mathrm{zeta}(0\cdot\mathrm{dtr},\mathrm{Hant},\mathrm{Hion})\cdot\mathrm{rtd} = 86.357$$

$$\mathrm{Gb}(T,\mathrm{Hb},\mathrm{Hion}) := \frac{G(T,\mathrm{Hb},\mathrm{Hion})}{2} - p(T,\mathrm{Hb},\mathrm{Hion})\cdot\sin\left(\frac{\mathrm{zeta}(T,\mathrm{Hb},\mathrm{Hion})}{3}\right)$$

$$\mathrm{Gi}(T,\mathrm{Hb},\mathrm{Hion}) := G(T,\mathrm{Hb},\mathrm{Hion}) - \mathrm{Gb}(T,\mathrm{Hb},\mathrm{Hion})$$

The reflection angle from the ionosphere is Tion=2πG/(2πRe)=G/Re

$$\mathrm{Tc} := 2.7\cdot\mathrm{dtr} \qquad \mathrm{Hion} = 1\times 10^5$$

$$\mathrm{Gc} := G(\mathrm{Tc},\mathrm{Hant},\mathrm{Hion}) \qquad \mathrm{Gc} = 955211.291$$

$$\mathrm{Gci} := \mathrm{Gi}(\mathrm{Tc},\mathrm{Hant},\mathrm{Hion}) \qquad \mathrm{Gci} = 953161.727$$

$$\mathrm{Gcb} := \mathrm{Gb}(\mathrm{Tc},\mathrm{Hant},\mathrm{Hion}) \qquad \mathrm{Gcb} = 2049.564$$

$$\mathrm{Tion}(T,\mathrm{Hb},\mathrm{Hion}) := \frac{G(T,\mathrm{Hb},\mathrm{Hion})}{\mathrm{Re}} + T$$

$$\mathrm{Tion}(\mathrm{Tc},\mathrm{Hant},\mathrm{Hion})\cdot\mathrm{rtd} = 11.29$$

Sanity check; answer is relative error: $\frac{8724000}{Gc \cdot 2} = 4.567$

$$\frac{2 \cdot Gcb^3 - 3 \cdot Gc \cdot Gcb^2 + \left[Gc^2 - 2 \cdot Ae \cdot (Hion + Hant)\right] \cdot Gcb + 2 \cdot Ae \cdot Hant \cdot Gc}{2 \cdot Ae \cdot Hant \cdot Gc} = -1.73432 \times 10^{-13}$$

From which:

$$\gamma b(T, Hb, Hion) := \frac{Gb(T, Hb, Hion)}{Ae}$$

$$Rfb(T, Hb, Hion) := \sqrt{(Ae + Hb)^2 + Ae^2 - 2 \cdot Ae \cdot (Ae + Hb) \cdot \cos(\gamma b(T, Hb, Hion))}$$

$$\gamma i(T, Hb, Hion) := \gamma(T, Hb, Hion) - \gamma b(T, Hb, Hion)$$

$$Rfi(T, Hb, Hion) := \sqrt{(Ae + Hion)^2 + Ae^2 - 2 \cdot Ae \cdot (Ae + Hion) \cdot \cos(\gamma i(T, Hb, Hion))}$$

$\gamma i(0, Hant, Hion) \cdot rtd = 8.61$

$\gamma b(0, Hant, Hion) \cdot rtd = 0.145$

$Rfb(0, Hant, Hion) = 21452.77$

$Rfi(0, Hant, Hion) = 1283535.619$

$DTr(0, Hant, Hion) = 1304988.006$

Incidence angle on the ground:

$ang := 89$

$$\psi(T, Hb, Hion) := \operatorname{acos}\left[(Ae + Hb) \cdot \frac{\sin(\gamma b(T, Hb, Hion))}{Rfb(T, Hb, Hion)}\right]$$

$\psi(ang \cdot dtr, Hant, Hion) \cdot rtd = 89.014$

Path difference, one approach:

$$Diff1(T, Hb, Hion) := \frac{4 \cdot Rfi(T, Hb, Hion) \cdot Rfb(T, Hb, Hion) \cdot \sin(\psi(T, Hb, Hion))^2}{DTr(T, Hb, Hion) + Rfi(T, Hb, Hion) + Rfb(T, Hb, Hion)}$$

More directly: $Diff(T, Hb, Hion) := Rfb(T, Hb, Hion) + Rfi(T, Hb, Hion) - DTr(T, Hb, Hion)$

$ang := 88.$ $\quad Hant = 100$ $\quad Diff(ang \cdot dtr, Hant, Hion) = 199.878$

$Diff1(ang \cdot dtr, Hant, Hion) = 199.881$

Wave divergence factor is:

$$Div(T, Hb, Hion) := \left(\sqrt{1 + \frac{2 \cdot Gi(T, Hb, Hion) \cdot Gb(T, Hb, Hion)}{Ae \cdot G(T, Hb, Hion) \cdot \sin(\psi(T, Hb, Hion))}}\right)^{-1}$$

Plane earth Incidence angle, redefine the variable, note that θ and T are nearly the same: $\theta(T, Hb, Hion) := \psi(T, Hb, Hion)$

The dielectric constant of the earth is: $\varepsilon(\varepsilon_r, \sigma, fMHz) := \varepsilon_r - \frac{0.2 \cdot c^2 \cdot \sigma}{fMHz} \cdot j$

and $x_H(\theta, \varepsilon) := \sqrt{\varepsilon - \cos(\theta)^2}$ or $x_V(\theta, \varepsilon) := \frac{x_H(\theta, \varepsilon)}{\varepsilon}$

"x" means xH or xV. $\Gamma s(\theta, x) := \frac{\sin(\theta) - x}{\sin(\theta) + x}$

Roughness factor:

$$Sr(fMHz, hrms, \theta) := e^{-2 \cdot \left(\frac{2 \cdot \pi \cdot fMHz}{c} \cdot hrms \cdot \sin(\theta)\right)^2} \cdot I0\left[2 \cdot \left(\frac{2 \cdot \pi \cdot fMHz}{c} \cdot hrms \cdot \sin(\theta)\right)^2\right]$$

The ground reflection coefficient with roughness:

$$\Gamma_H(\theta, \varepsilon, fMHz, hrms) := \Gamma s(\theta, x_H(\theta, \varepsilon)) \cdot Sr(fMHz, hrms, \theta)$$

$$\Gamma_V(\theta, \varepsilon, fMHz, hrms) := \Gamma s(\theta, x_V(\theta, \varepsilon)) \cdot Sr(fMHz, hrms, \theta)$$

The phase difference ϕ between the direct and the ground reflected rays is:

$$Sr(50, 2, 5 \cdot dtr) = 0.936$$

$$\phi(T, Hb, Hion, fMHz) := \frac{2 \cdot \pi \cdot fMHz}{c} \cdot Diff(T, Hb, Hion)$$

$$t := 1$$

$$\Gamma_H(t, 12, 100, 0) = -0.605$$

Phasor, divergence, and path ratio multiplier (not used, equals 1) terms:

$$ex(T, Hb, Hi, F) := e^{-j \cdot \phi(T, Hb, Hi, F)} \cdot Div(T, Hb, Hi)$$

$$PathRatio := \left(1 + \frac{Diff(T, Hb, Hi)}{DTr(T, Hb, Hi)}\right)^{-1}$$

Note that θ and T are nearly the same, so antenna elevation pattern is not a big factor.

This version omits the surface wave, but includes both polarizations added as powers, and mixed in a Horizontal power to Vertical power ratio: HV=H/V ... "0" = pure vertical; "infinity" = pure horizontal ... hr the rms ground surface roughness, and lowers the effective antenna height by hrms/2.

T radians; Hb, Hi, hrms m; ε complex; F MHz

$$P(T, Hb, Hi, \varepsilon, hr, F, HV) := \frac{\left[HV \cdot \left(\left|1 + ex\left(T, Hb - \frac{hr}{2}, Hi, F\right) \cdot \Gamma_H\left(\theta\left(T, Hb - \frac{hr}{2}, Hi\right), \varepsilon, F, hr\right)\right|\right)^2 + \left(\left|1 + ex\left(T, Hb - \frac{hr}{2}, Hi, F\right) \cdot \Gamma_V\left(\theta\left(T, Hb - \frac{hr}{2}, Hi\right), \varepsilon, F, hr\right)\right|\right)^2\right]}{1 + HV}$$

Free space loss, d m; F MHz: $Pf(d, F) := \left(\frac{c}{4 \cdot \pi \cdot F \cdot d}\right)^2$ $10 \cdot \log(Pf(1, 51)) = -6.599$

Free space loss for the distance of one ionospheric bounce is:

$$\mathrm{Pfs}(T, Hb, Hion, FMHz) := 20 \cdot \log\left[\frac{c}{4 \cdot \pi \cdot FMHz \cdot (2 \cdot \mathrm{DTr}(T, Hb, Hion))}\right] \qquad 20 \cdot \log\left(\frac{c}{4 \cdot \pi}\right) = 27.552$$

$$ang := 6.899$$

$$Hant = 100$$

decibels:

$$\mathrm{Pfs}(ang \cdot dtr, 32, Hion \cdot 2.5, 14) = -123.615 \qquad \mathrm{DTr}(ang \cdot dtr, Hant, Hion) = 6.372 \times 10^{5}$$

Reflection loss from ground is:

$$\mathrm{Refl}(T, Hb, Hi, \varepsilon, hr, F, HV) := \frac{\left[\begin{array}{l} HV \cdot \left(\left|\mathrm{ex}\left(T, Hb - \frac{hr}{2}, Hi, F\right) \cdot \Gamma_H\left(\theta\left(T, Hb - \frac{hr}{2}, Hi\right), \varepsilon, F, hr\right)\right|\right)^2 \ldots \\ + \left(\left|\mathrm{ex}\left(T, Hb - \frac{hr}{2}, Hi, F\right) \cdot \Gamma_V\left(\theta\left(T, Hb - \frac{hr}{2}, Hi\right), \varepsilon, F, hr\right)\right|\right)^2 \end{array}\right]}{1 + HV}$$

$$\mathrm{Rcoef}(T, Hb, Hi, \varepsilon, hr, F, HV) := 10 \cdot \log(\mathrm{Refl}(T, Hb, Hi, \varepsilon, hr, F, HV))$$

$$\varepsilon Sea(fMHz) := \varepsilon(70.6, 4.54, fMHz) \qquad \varepsilon dry(f) := \varepsilon(5.0, 0.005, f) \qquad \varepsilon med(f) := \varepsilon(12, 0.05, f)$$

Nia := 100 ia := 0 .. Nia $a_{ia} := \left(ia \cdot dtr \cdot \frac{23.5}{Nia}\right) + dtr$ HV := 0 Hb := 10

Fr := 14.1 roughness parameter hr := 0.0 Fr = 14.1

$Rcoef\left(a_{ia}, Hb, Hion \cdot 2.5, \varepsilon Sea(Fr), hr, Fr, 0.001\right)$

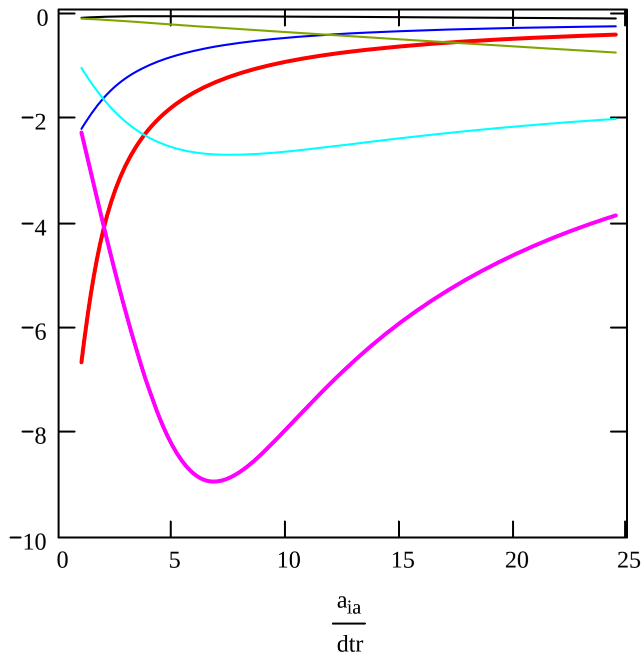
$Rcoef\left(a_{ia}, Hb, Hion \cdot 2.5, \varepsilon Sea(Fr), hr, Fr, 1\right)$

$Rcoef\left(a_{ia}, Hb, Hion \cdot 2.5, \varepsilon Sea(Fr), hr, Fr, 100\right)$

$Rcoef\left(a_{ia}, Hb, Hion \cdot 2.5, \varepsilon med(Fr), hr, Fr, 0.001\right)$

$Rcoef\left(a_{ia}, Hb, Hion \cdot 2.5, \varepsilon med(Fr), hr, Fr, 1\right)$

$Rcoef\left(a_{ia}, Hb, Hion \cdot 2.5, \varepsilon med(Fr), hr, Fr, 100\right)$



Fr = 14.1 roughness parameter hr2 := 3

$Rcoef\left(a_{ia}, Hb, Hion \cdot 2.5, \varepsilon Sea(Fr), hr2, Fr, 0.001\right)$

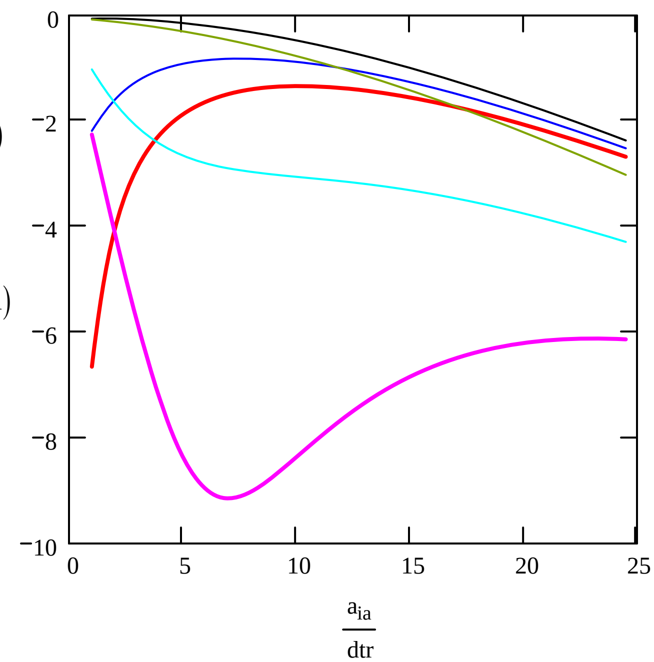
$Rcoef\left(a_{ia}, Hb, Hion \cdot 2.5, \varepsilon Sea(Fr), hr2, Fr, 1\right)$

$Rcoef\left(a_{ia}, Hb, Hion \cdot 2.5, \varepsilon Sea(Fr), hr2, Fr, 100\right)$

$Rcoef\left(a_{ia}, Hb, Hion \cdot 2.5, \varepsilon med(Fr), hr2, Fr, 0.001\right)$

$Rcoef\left(a_{ia}, Hb, Hion \cdot 2.5, \varepsilon med(Fr), hr2, Fr, 1\right)$

$Rcoef\left(a_{ia}, Hb, Hion \cdot 2.5, \varepsilon med(Fr), hr2, Fr, 100\right)$



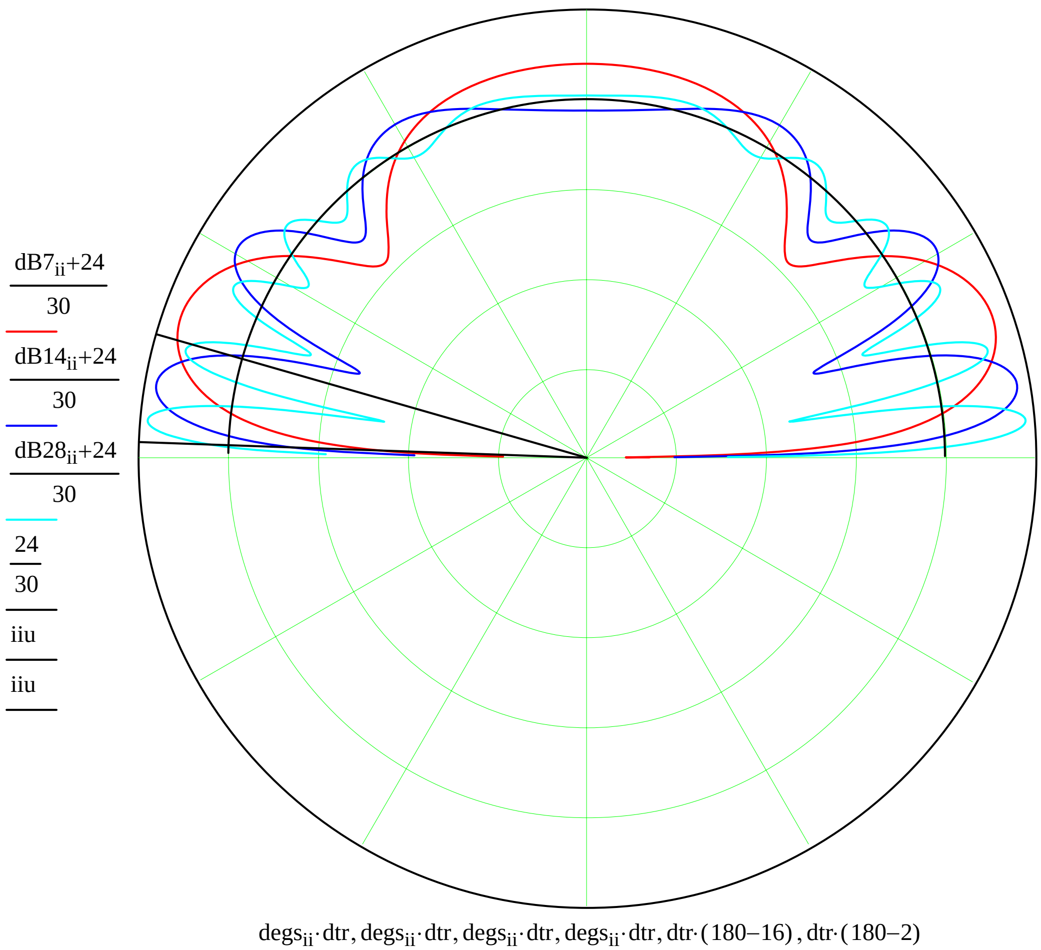
Traditional transmit view; $Hiono := Hion$ $Hbase := 32$ $HVratio := 20$ $\varepsilon p := 10$

$Ndeg := 1000$ $ii := 0 .. Ndeg$ $degs_{ii} := ii \cdot \frac{179.}{Ndeg} + 0.25$ $iiu := 0 .. 1$

$$dB7_{ii} := 10 \cdot \log\left(\left|P\left(degs_{ii} \cdot dtr, Hbase, Hiono, \varepsilon p, hrms, 7, HVratio\right)\right|\right)$$

$$dB14_{ii} := 10 \cdot \log\left(\left|P\left(degs_{ii} \cdot dtr, Hbase, Hiono, \varepsilon p, hrms, 14, HVratio\right)\right|\right)$$

$$dB28_{ii} := 10 \cdot \log\left(\left|P\left(degs_{ii} \cdot dtr, Hbase, Hiono, \varepsilon p, hrms, 28, HVratio\right)\right|\right)$$



$Hbase = 32$

$HbaseA := 15$

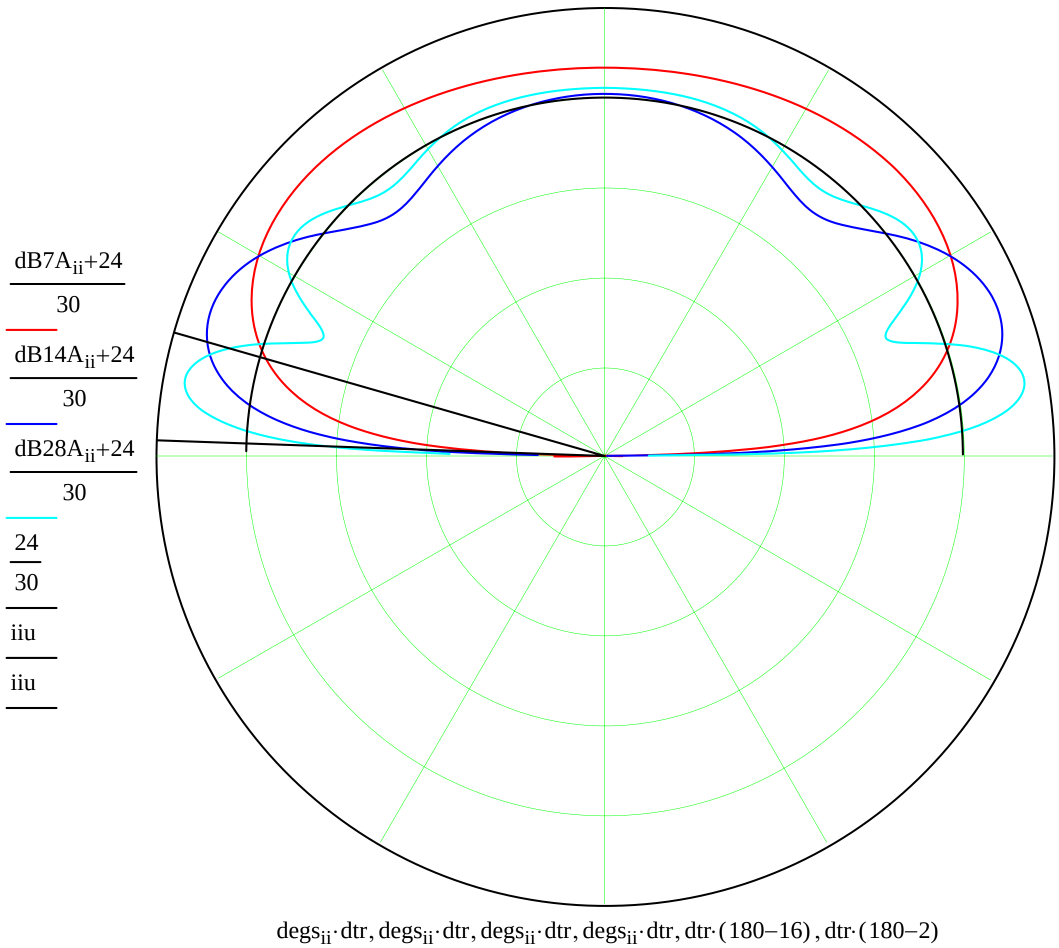
$dB7A_{ii} := 10 \cdot \log\left(\left|P\left(degs_{ii} \cdot dtr, HbaseA, Hiono, \varepsilon p, hrms, 7, HVratio\right)\right|\right)$

$dB14A_{ii} := 10 \cdot \log\left(\left|P\left(degs_{ii} \cdot dtr, HbaseA, Hiono, \varepsilon p, hrms, 14, HVratio\right)\right|\right)$

$degs_{165} = 29.785$

$dB28A_{ii} := 10 \cdot \log\left(\left|P\left(degs_{ii} \cdot dtr, HbaseA, Hiono, \varepsilon p, hrms, 28, HVratio\right)\right|\right)$

$dB7A_{165} = 2.74$



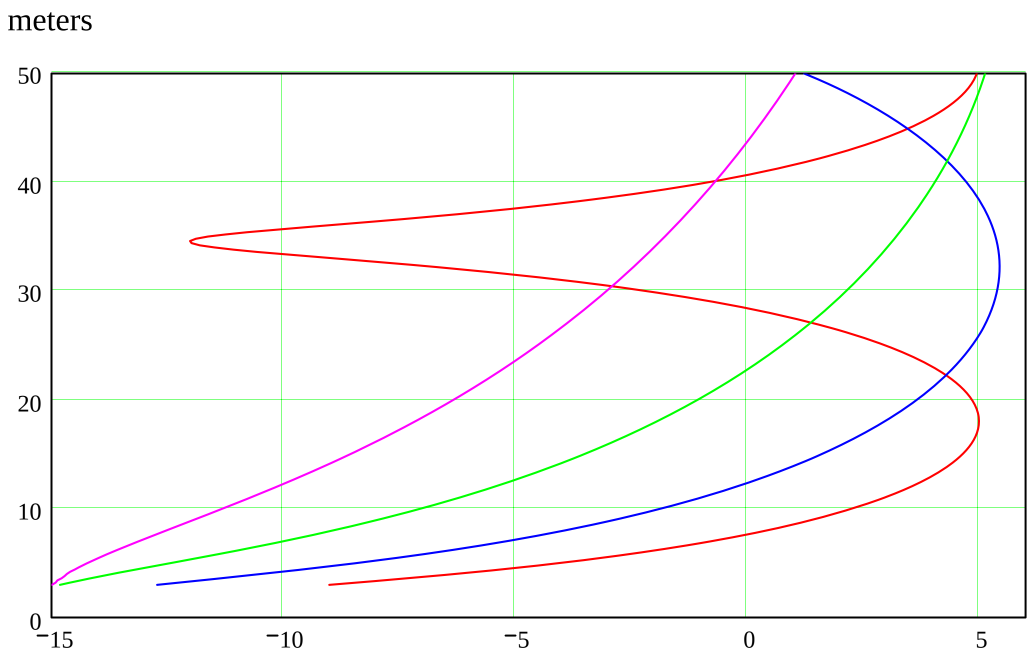
$HbaseA = 15$

This is the function and parameters used for detailed study:

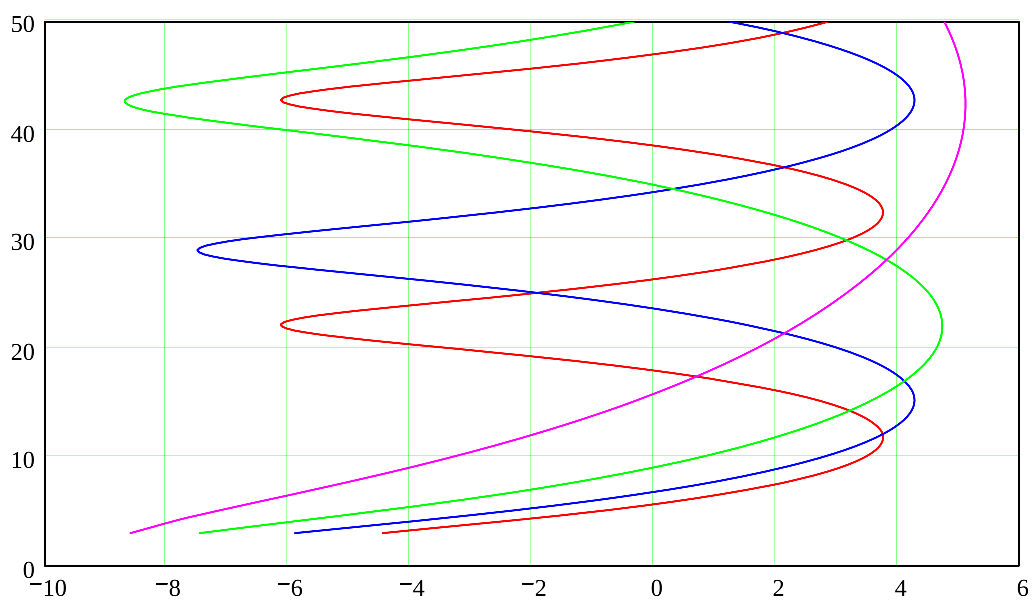
$t0 := 5$ deg take off angle $HV := 10$ $hrms = 3$ $Hion = 1 \times 10^{5}$ $Hi := Hion$

$h := hrms, hrms + .2 .. 50$

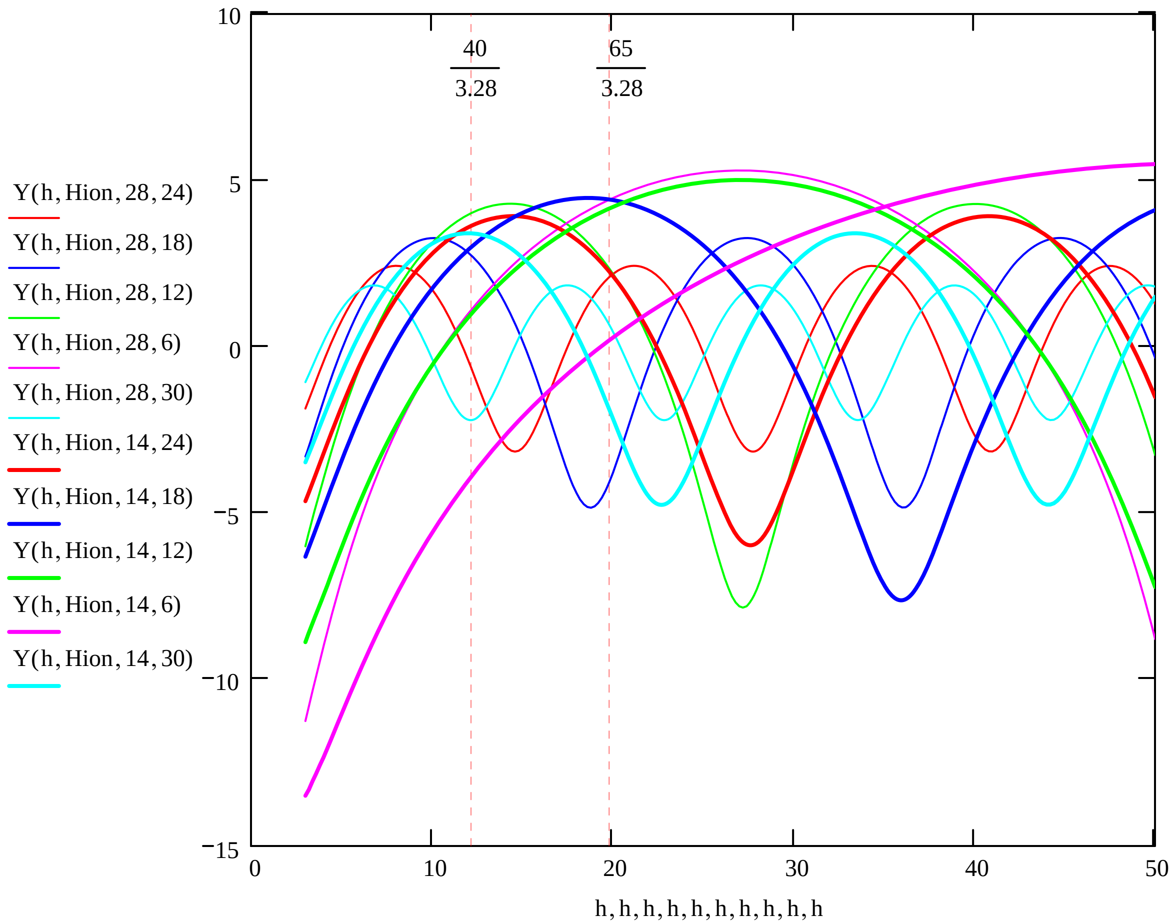
$$Y(h, H, fMHz, Tdeg) := 10 \cdot \log\left(P\left(Tdeg \cdot dtr, h, H, \varepsilon(12, 0.005, fMHz), hrms, fMHz, HV\right)\right)$$



$to := 15$ deg take off angle



gain, dB for different takeoff angles at 14 and 28 MHz:



Adjust the propagation for antenna pattern vs. elevation angle:

Choose a half power BW angular extent: $BW3 := 16$

Antenna directivity is 10 dBi; E and H plane patterns equal, +/- 28 deg half power BW, similar to a 4-elt yagi: $Gain(BW) := \frac{32400}{BW^2}$

2-elt is +/- 38 deg 7 dBi

Yagi gain is omitted, but equals $YagidB := 10 \cdot \log(Gain(2 \cdot BW3))$ $YagidB = 15.002$

Cosine patterm equivalent: $n := 3$ $Npower := root\left[\left(\left|\cos(BW3 \cdot dtr)\right|\right)^n - 0.5, n\right]$ $Npower = 17.544$

$PL(TO) := (10 \cdot Npower \cdot \log(\cos(TO \cdot dtr)))$ $PL(BW3) = -3.01$

$$Z(h, Hion, F, TO) := Y(h, Hion, F, TO) + PL(TO)$$